A Strip Method for Prediction of Damping in Subsonic Wind Tunnel and Flight Flutter Tests

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A strip method has been developed for the prediction of subcritical damping characteristics for guidance of subsonic wind tunnel and flight flutter tests. The transient aerodynamic coefficients are found from a Fourier transform of the two-dimensional incompressible oscillatory coefficients. A series of transient aerodynamic influence coefficients (AIC's) has been derived including a newly defined matrix of aerodynamic lag AIC's. Compressibility, sweep and finite span effects are included by varying the lift curve slope and aerodynamic center locations on each strip in accord with static wind tunnel data. A collocation formulation of the subcritical flutter eigenvalue problem is presented. A special technique for solving the complex eigenvalue problem is reviewed and illustrated in an example of a restrained (cantilevered) wing with five strips and ten elastic degrees of freedom. Finally, the simplifications that result from reducing the size of the eigenvalue problem by modal methods are discussed.

Nomenclature

a = element of flexibility matrix; amplitude of flexible or rigid modes in modal formulation
B_n = element of aerodynamic lag function matrix; subscript indicates correspondence with nth term in exponential approximation to the Wagner function
b = semichord of wing strip; b_r is reference value
C = element of damping matrix for unrestrained vehicle; $C(k)$ is the Theodorsen function
C_{hB} = element of lag AIC matrix
C_{hDh} = element of damping AIC matrix
C_{hD^2h} = element of inertial AIC matrix
C_{hs} = element of static AIC matrix
$\bar{c}$ = reference chord
$c_{l\alpha}$ = lift curve slope of wing strip
$D(t)$ = Duhamel integral of circulation and Wagner function
F = collocation control point force
$\mathfrak{F}$ = Fourier transform operator

$$\mathfrak{F}f(t) = \int_{-\infty}^{+\infty} f(t)\exp(-i\omega t)dt$$

g = artificial required structural damping coefficient in conventional flutter solution
H = element of deflection interpolation matrix
h = deflection of $\frac{1}{4}$ chord point of uncambered strip
h_1, h_2, h_3 = deflections of collocation control points on cambered strip
h_R = element of rigid body modal matrix
I = element of unit matrix
K = element of stiffness matrix for unrestrained vehicle
k = reduced frequency of oscillatory motion, $k = \omega b/V$
L = normal force on wing strip
$\mathfrak{L}$ = Laplace transform operator

$$\mathfrak{L}f(t) = \int_0^\infty f(t)\exp(-st)dt$$

M = pitching moment of wing strip; element of mass matrix
N = generalized camber force; number of modes in modal solution
Q = quantity proportional to circulation about airfoil
q = dynamic pressure
r = $c_{l\alpha}/2\pi \cos\Lambda + 2\xi - \frac{1}{2}$
S = reference planform area
s = Laplace transform parameter
t, τ = time, and its dummy variable
V = freestream velocity
W = element of differentiation matrix
w = $\dot{h}$
Δy = width of wing strip
α = angle of attack
α_1, α_2 = coefficients in two-term exponential approximation to the Wagner function
β_1, β_2 = coefficients in exponent in two-term exponential approximation to the Wagner function
γ = eigenvalue, coefficient in exponential time dependence, $\exp(\gamma t)$
γ_s = system decay coefficient
λ = modified eigenvalue, $\lambda = 1/(\gamma_0 - \gamma)$, where γ_0 is a shifted value of γ
ζ = amplitude of camber deflection
ξ = aerodynamic center location as a fraction of the chord
Λ = sweep angle of $\frac{1}{4}$ chord line
ρ = air density
$\Phi(Vt/b)$ = Wagner function
ω = circular frequency

Subscripts and Superscripts

c = corrected for effects of compressibility, sweep and finite span
$(\dot{\ })$ = $d(\ )/dt$
$(\ )'$ = $d(\ )/d\tau$; also denotes that aerodynamic effects are included with corresponding mechanical terms

Matrix Notation

$[\ ]$ = rectangular or square
$[\ ]^{-1}$ = inverse
$[\ ]^T$ = transpose
$\lceil\ \rfloor$ = diagonal
$\{\ \}$ = column
$\lfloor\ \rfloor$ = row
$[\ ]^E$ = matrix expansion operation defined in Eq. (78)

Received September 21, 1967; revision received August 19, 1968. The authors wish to acknowledge E. C. Yates Jr. of NASA Langley Research Center, Hampton, Va., and F. J. Frueh of Acronautics R & D Inc., Burbank, Calif., for reviewing the preliminary manuscript and for their helpful suggestions. Work presented herein was conducted by the Douglas Aircraft Division under company-sponsored research and development funds.

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Introduction

THE conventional method of flutter analysis was determined by a limited knowledge of unsteady aerodynamic loads: more extensive theoretical solutions were available for the case of steady-state harmonic motion than for the more general case of transient motion because of the relative simplicity of the mathematical formulation. For this reason, it was necessary to assume harmonic motion and to seek the values of speed and frequency for which this actually was the case. To facilitate this approach, the mathematical concept of a required artificial damping, which must be added to or subtracted from the system to sustain the assumed harmonic motion, was introduced. The flutter speed and frequency were then determined from the conditions under which the required artificial damping is zero.

We therefore see the peculiar nature of the stability analysis of flutter. It is a completely specialized technique having no relationship to the conventional stability methods for transient systems, e.g., Nyquist plots, root-locus plots, etc. In particular, the required artificial damping can be no more than a *qualitative* measure of stability. It is a mathematical artifice used to seek out the flutter point, but it cannot be interpreted as having a physical significance as a measure, e.g., of the decay rate to be observed in a subcritical flight flutter test. A transient formulation of the flutter problem is necessary to predict subcritical flight flutter response characteristics.

A number of studies[1-7] have attempted to relate the artificial damping to the actual damping in flight. Approximate solutions have been obtained for low subsonic and high supersonic flight regimes. However, in view of the stated limitations of these methods,[5] an alternate formulation is desirable to provide solutions when the methods of Refs. 1–7 fail.

In their survey of unsteady AIC's, Rodden and Revell[8] offered some tentative suggestions on various formats for representing transient AIC's. A specific format has been obtained for a subsonic strip theory and is presented in this paper. From this and the equations of motion, the subcritical flutter frequencies and dampings follow as a routine eigenvalue problem. This development is related to the method proposed by Richardson[9] and may be regarded as an extension for the specific case of strip theory. It should be noted that solutions to the flutter problem on analog computers using strip theory have always utilized a transient formulation similar to the present one. The present formulation may be regarded as a generalization of the formulation for an analog computer for use on a digital computer.

The present method is applicable to surfaces with moderate to high aspect ratios, since the aerodynamic strip theory approximation is inaccurate if the surface aspect ratio is low. For unswept surfaces with moderate to high aspect ratio, streamwise deformation may usually be neglected and the aerodynamic degrees of freedom of plunging and pitching motions are sufficient to specify the aerodynamic loading. On a swept surface, however, the streamwise deformation depends on the orientation of the internal chordwise stiffening. If the bulkheads (ribs) are aligned in the streamwise direction, then, again, streamwise deformation may be neglected and no aerodynamic loads arise from cambering motion. If the bulkheads are aligned normal to the swept spars, then cross sections normal to the spars do not deform but induce a camber in the streamwise direction. Aircraft wing construction typically utilizes both orientations: bulkheads normal to the spars in general, but streamwise in regions of engines and store pylons. This paper therefore presents a streamwise strip theory with three aerodynamic degrees of freedom, plunging, pitching, and cambering.

Transient Aerodynamic Loads from Arbitrary Motion

The oscillatory aerodynamic coefficients for the three degrees of freedom of plunging, pitching, and parabolic cambering have been given by Spielberg.[10] Since the oscillatory loading may be regarded as the Fourier transform of the transient loading, the transient loads required for the present analysis are given by the inverse Fourier transform of Spielberg's results. We rewrite Spielberg's expressions for lift, first moment about the $\frac{1}{4}$ chord, and generalized camber force,‡ respectively, as follows:

$$L(\omega) = \pi\rho b^2[\omega^2 h + (\omega^2/2 - i\omega V/b)b\alpha + \tfrac{3}{4}\omega^2\zeta] + 2\pi\rho Vb[C(k)/i\omega][\omega^2 h + (\omega^2 - i\omega V/b)b\alpha + (\omega^2/2 + i\omega V/b)\zeta] \quad (1)$$

$$M(\omega) = \pi\rho b^3[\tfrac{1}{2}\omega^2 h + (3\omega^2/8 - i\omega V/b)b\alpha + (3\omega^2/8 + i\omega V/2b + V^2/b^2)\zeta] \quad (2)$$

$$N(\omega) = \pi\rho b^2[\tfrac{3}{4}\omega^2 h + (3\omega^2/8 - i\omega V/b)b\alpha + (7\omega^2/12 + V^2/2b^2)\zeta] + \pi\rho Vb[C(k)/i\omega] \times [\omega^2 h + (\omega^2 - i\omega V/b)b\alpha + (\omega^2/2 + i\omega V/b)\zeta] \quad (3)$$

The inverse Fourier transforms of Eqs. (1–3) are easily found. The inverse transforms of the terms involving the Theodorsen function $C(k)$ are found from the Convolution Theorem for Fourier transforms after noting that the Fourier transform of the Wagner function[11,12] is§

$$\mathfrak{F}\Phi(Vt/b) = C(k)/i\omega \quad (4)$$

The transient loads therefore become

$$L(t) = -\pi\rho b^2[\ddot{h} + \tfrac{1}{2}b\ddot{\alpha} + V\dot{\alpha} + \tfrac{3}{4}\ddot{\zeta}] - 2\pi\rho VbD(t) \quad (5)$$

$$M(t) = -\pi\rho b^3[\tfrac{1}{2}\ddot{h} + \tfrac{3}{8}b\ddot{\alpha} + V\dot{\alpha} + \tfrac{3}{8}\ddot{\zeta} - (V/2b)\dot{\zeta} - (V^2/b^2)\zeta] \quad (6)$$

$$N(t) = -\pi\rho b^2[\tfrac{3}{4}\ddot{h} + \tfrac{3}{8}b\ddot{\alpha} + V\dot{\alpha} + \tfrac{7}{12}\ddot{\zeta} - (V^2/2b^2)\zeta] - \pi\rho VbD(t) \quad (7)$$

where the Duhamel integral $D(t)$ is

$$D(t) = \int_0^t \Phi\left[\frac{V(t-\tau)}{b}\right] Q'(\tau)d\tau \quad (8)$$

in which

$$Q'(\tau) = dQ(\tau)/d\tau \quad (9a)$$

$$= h'' + b\alpha'' + V\alpha' + \tfrac{1}{2}\zeta'' - (V/b)\zeta' \quad (9b)$$

and we have assumed the system to be initially at rest. The quantity $Q(t)$ may be interpreted as the downwash at the $\frac{3}{4}$ chord location in the case of plunging and pitching of a rigid airfoil. However, it actually measures the circulation about the airfoil and, in general, does not lend itself to a simple geometric interpretation.

We now consider evaluation of the Duhamel integral. An explicit expression for the Wagner function does not exist and it can only be determined numerically. A number of analytical approximations have been proposed. Of particular interest at subsonic speeds is the exponential approximation in the form

$$\Phi(Vt/b) = 1 - \Sigma\alpha_n \exp(-\beta_n Vt/b) \quad (10)$$

For example, W. P. Jones[13] has suggested a two-term approximation in which $\alpha_1 = 0.165$, $\beta_1 = 0.041$, $\alpha_2 = 0.335$, and $\beta_2 = 0.320$. Equation (4) and the Fourier transform of Eq. (10) provide a general equation for determining the coefficients in an exponential approximation.

$$1 - \Sigma\alpha_n/(1 - i\beta_n/k) = C(k) \quad (11)$$

‡ Spielberg incorrectly describes the generalized camber force as a second moment about the midchord. It is actually equal to the lift force minus the second moment about the midchord divided by b^2.

§ Note that the transform variable here is ω rather than k as in Ref. 12.

This expression permits finding an approximate solution to the transient aerodynamic problem when the only solution available is for the oscillatory case. Although Eq. (11) was obtained entirely from incompressible flow considerations, it may be used to determine the α_n and β_n for subsonic compressible flow if appropriate values of $C(k)$ are used for Mach numbers other than zero. In this way the coefficients α_n and β_n are seen to be functions of the subsonic Mach number.

Substituting the approximation of Eq. (10) into the Duhamel integral leads to

$$D(t) = Q(t) - \Sigma\alpha_n B_n(t) \tag{12}$$

where the functions $B_n(t)$ are given by

$$B_n(t) = \exp\left(-\beta_n \frac{Vt}{b}\right) \int_0^t Q'(\tau) \exp\left(\beta_n \frac{V\tau}{b}\right) d\tau \tag{13}$$

The functions $B_n(t)$ measure the lag in the induced aerodynamic loads in following the motion. By differentiating both sides of Eq. (13), a differential equation that is equivalent to the Duhamel integral is found;

$$\dot{B}_n + (\beta_n V/b)B_n = \dot{Q}(t) \tag{14}$$

This is an extremely practical result. It eliminates the need of evaluating Duhamel integrals but replaces them by equivalent aerodynamic equations of motion. In a response analysis, the mechanical equations of motion can be written as an equivalent system of simultaneous first-order differential equations, so that an aeroelastic response calculation becomes a problem of numerical integration of simultaneous mechanical and aerodynamic first-order differential equations. A consistent method of solution can be employed, e.g., the Adams-Moulton method, leading to the same accuracy for the deflections and the aerodynamic lag functions $B_n(t)$. The necessity of special techniques for evaluating the Duhamel integrals has thereby been avoided. Equation (14) will permit formulation of the flutter problem as a more conventional stability analysis from which the actual damping to be observed in a subcritical wind tunnel or flight flutter test can be estimated.

The final consideration in this section is the extension of the foregoing results to the three-dimensional case by means of strip theory. We have seen that the loading on a strip depends on the linear and angular deflections, velocities, and accelerations and on the aerodynamic lag functions. It is therefore possible to define a series of AIC's that are compatible with these observations. In this way incompressible strip theory leads to specific formats for the tentative recommendations of Ref. 8, Sec. IV. We define the transient AIC's by

$$\{F\} = \left(\frac{qS}{\bar{c}}\right)\left([C_{hs}]\{h\} + [C_{hDh}]\left\{\frac{\dot{h}\bar{c}}{V}\right\} + [C_{hD^2h}]\left\{\frac{\ddot{h}\bar{c}^2}{V^2}\right\} - \Sigma\alpha_n[C_{hB}]\left\{\frac{B_n\bar{c}}{V}\right\}\right) \tag{15}$$

where

$$\{\dot{B}_n\} + V\beta_n[1/b]\{B_n\} = \{\dot{Q}\} \tag{16a}$$

$$= [H]\{\ddot{h}\} + [W]\{\dot{h}\} \tag{16b}$$

We shall term the various matrices as follows $-[C_{hs}]$: the static AIC's; $[C_{hDh}]$: the damping AIC's; $[C_{hD^2h}]$: the inertial AIC's; $[C_{hB}]$: the lag AIC's; $[H]$: the interpolation matrix; and $[W]$: the differentiation¶ matrix. We shall discuss the format that each of these matrices assumes with strip theory later. First, we must recognize that a number of modifications are necessary in order to apply incompressible results to arbitrary subsonic speeds. These modifications are discussed in the next section.

¶ The terminology for $[H]$ and $[W]$ as interpolation and differentiation matrices, respectively, stems from the interpretation of the quantity $Q(t)$ as the downwash of the $\frac{3}{4}$ chord location in the case of a rigid chord, since the downwash is found from interpolation and differentiation of the deflections. We apply the same terminology to the general case by analogy.

Empirical Modification of Theoretical Aerodynamic Results

In a recent paper, Yates[14] has demonstrated the rather remarkable success of a modified strip theory for flutter predictions at speeds from subsonic to hypersonic. His correlations were achieved on a broad range of swept and unswept wings of moderate to high aspect ratio. We wish therefore to avail ourselves of his modification procedures in accounting for three-dimensional and Mach number effects.

Yates based his modifications on values of sectional lift-curve slope $c_{l\alpha}$ and aerodynamic center location ξ (measured as a fraction of the local chord aft of the leading edge), which he assumed were available from any suitable steady-flow aerodynamic theory or from measured load distributions. The modifications were applied to the circulatory aerodynamic terms but not to the noncirculatory terms.

We apply Yates' modifications as directly as possible to Eqs. (5–8), noting, however, that he does not treat camber per se. We simply adjust the camber terms by modifying the section lift-curve slope. Following Yates, then, we correct Eqs. (5–8) for the effects of sweep, aspect ratio, and Mach number by writing

$$L_c(t) = -\pi\rho b^2(\ddot{h} + \tfrac{1}{2}b\ddot{\alpha} + V\dot{\alpha} + \tfrac{3}{4}\ddot{\zeta}) - c_{l\alpha}\rho VbD_c(t) \tag{17}$$

$$M_c(t) = -\pi\rho b^3[\tfrac{1}{2}\ddot{h} + \tfrac{3}{8}b\ddot{\alpha} + V\dot{\alpha}(c_{l\alpha}/2\pi \cos\Lambda + 2\xi - \tfrac{1}{2}) + \tfrac{3}{8}\ddot{\zeta} - (V/2b)\dot{\zeta} - (V^2/b^2)\zeta] - c_{l\alpha}(2\xi - \tfrac{1}{2})\rho Vb^2D_c(t) \tag{18}$$

$$N_c(t) = -\pi\rho b^2[\tfrac{3}{4}\ddot{h} + \tfrac{3}{8}b\ddot{\alpha} + V\dot{\alpha} + \tfrac{7}{12}\ddot{\zeta} - (V^2/2b^2)\zeta] - (c_{l\alpha}/2)\rho VbD_c(t) \tag{19}$$

where

$$D_c(t) = Q_c(t) - \Sigma\alpha_n B_n(t) \tag{20}$$

in which

$$Q_c(t) = \dot{h} + b\dot{\alpha}(c_{l\alpha}/2\pi \cos\Lambda + 2\xi - \tfrac{1}{2}) + V\alpha + \tfrac{1}{2}\dot{\zeta} - (V/b)\zeta \tag{21}$$

and now

$$B_n(t) = \exp(-\beta_n Vt/b) \int_0^t Q'_c(\tau) \exp(\beta_n V\tau/b)d\tau \tag{22}$$

In the foregoing, we have used the approximation that the theoretical two-dimensional incompressible lift-curve slope of a swept wing is $2\pi \cos\Lambda$. With these modified expressions for the aerodynamic loads, we are now in a position to derive the AIC's.

The AIC's from Strip Theory

In the case of strip theory, the matrix of AIC's appears in a partitioned form. For example, the AIC's for a two-strip wing appear as

$$[C_h] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{h1} & 0 \\ 0 & 0 & C_{h2} \end{bmatrix} \tag{23}$$

where the C_{hi} are the AIC's for each strip and the null partition has been reserved for degrees of freedom for which the aerodynamic forces might be neglected, e.g., an engine or fuselage, or obtained from a different aerodynamic theory, e.g., slender body theory for the fuselage. Hence, it will only be necessary to form each of the four classes of transient

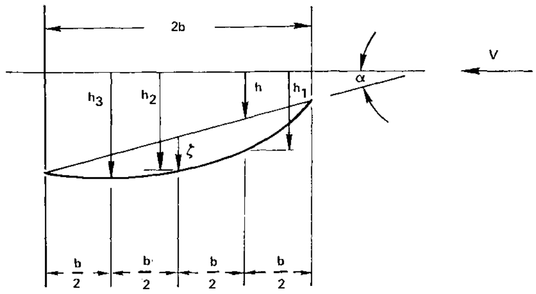

Fig. 1 Geometry of cambering airfoil.

AIC's plus the interpolation and differentiation matrices for a single strip.

The AIC's as defined in Eq. (15) require developing three matrix relations. The aerodynamic loads must be related to the aerodynamic displacements h, α, and ζ. The control point forces must be related to the aerodynamic loads. Finally, the aerodynamic displacements must be related to the control point deflections. We find it convenient to locate the forward control point at the $\frac{1}{4}$ chord location, the middle control point at midchord, and the aft control point at the $\frac{3}{4}$ chord location. In general, the choice of control point locations should be such that the parabolic camber curve is fitted accurately. Our present choice is made only for simplicity in the following development.

The relationship between the aerodynamic loads and the aerodynamic displacements is obtained by writing Eqs. (17–21) in matrix form and adding the spanwise factor of Δy, the width of the strip.

$$\begin{Bmatrix} L_c \\ M_c/b \\ N_c \end{Bmatrix} = -\pi\rho b^2 \Delta y \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{4} & \frac{3}{8} & \frac{7}{12} \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ b\ddot{\alpha} \\ \ddot{\zeta} \end{Bmatrix} - \pi\rho V b \Delta y \begin{bmatrix} 0 & 1 & 0 \\ 0 & r & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{h} \\ b\dot{\alpha} \\ \dot{\zeta} \end{Bmatrix} + \pi\rho V^2 \Delta y \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} h \\ b\alpha \\ \zeta \end{Bmatrix} - c_{l\alpha}\rho V b \Delta y \begin{Bmatrix} 1 \\ 2\xi - \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix} D_c(t) \quad (24)$$

where

$$r = c_{l\alpha}/2\pi \cos\Lambda + 2\xi - \tfrac{1}{2} \quad (25)$$

and

$$D_c(t) = \lfloor 1 \quad r \quad \tfrac{1}{2} \rfloor \begin{Bmatrix} \dot{h} \\ b\dot{\alpha} \\ \dot{\zeta} \end{Bmatrix} + \left(\frac{V}{b}\right) \lfloor 0 \quad 1 \quad -1 \rfloor \begin{Bmatrix} h \\ b\alpha \\ \zeta \end{Bmatrix} - \Sigma \alpha_n B_n(t) \quad (26)$$

The AIC forces and the aerodynamic loads are related according to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & b/2 & b \\ \frac{3}{4} & 1 & \frac{3}{4} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} L_c \\ M_c \\ N_c \end{Bmatrix} \quad (27)$$

from which

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} \frac{5}{2} & -1 & -2 \\ -3 & 0 & 4 \\ \frac{3}{2} & 1 & -2 \end{bmatrix} \begin{Bmatrix} L_c \\ M_c/b \\ N_c \end{Bmatrix} \quad (28)$$

The relationship between the control point deflections and the aerodynamic deflections may be derived from Fig. 1 in terms of the deflections at the leading and trailing edges. Fitting a parabola through h_1, h_2, and h_3, we find the leading and trailing edge deflections to be

$$h_{le} = 3h_1 - 3h_2 + h_3 \quad (29)$$

and

$$h_{te} = h_1 - 3h_2 + 3h_3 \quad (30)$$

Then the aerodynamic deflections are

$$h = \tfrac{3}{4}h_{le} + \tfrac{1}{4}h_{te} \quad (31a)$$

$$= \tfrac{5}{2}h_1 - 3h_2 + \tfrac{3}{2}h_3 \quad (31b)$$

$$b\alpha = \tfrac{1}{2}(h_{te} - h_{le}) \quad (32a)$$

$$= -h_1 + h_3 \quad (32b)$$

$$\zeta = h_2 - \tfrac{1}{2}(h_{le} + h_{te}) \quad (33a)$$

$$= -2h_1 + 4h_2 - 2h_3 \quad (33b)$$

In matrix form the desired relationship is

$$\begin{Bmatrix} h \\ b\alpha \\ \zeta \end{Bmatrix} = \begin{bmatrix} \frac{5}{2} & -3 & \frac{3}{2} \\ -1 & 0 & 1 \\ -2 & 4 & -2 \end{bmatrix} \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix} \quad (34)$$

Note that this transformation matrix is the transpose of the force transformation matrix in Eq. (28).

The AIC's follow by combining Eqs. (24), (26), (28), and (34) and identifying the results with the corresponding definitions of the various AIC's in Eqs. (15) and (16). We choose the reference chord $\bar{c}$ as the reference semichord b_r. We find the static AIC's to be

$$[C_{hs}] = 2c_{l\alpha}\left(b_r \frac{\Delta y}{S}\right) \begin{bmatrix} 2\xi - 2 & -8\xi + 8 & 6\xi - 6 \\ 1 & -4 & 3 \\ -2\xi & 8\xi & -6\xi \end{bmatrix} + 8\pi \left(b_r \frac{\Delta y}{S}\right) \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad (35)$$

The damping AIC's are

$$[C_{hDh}] = c_{l\alpha}\left(b \frac{\Delta y}{S}\right) \times \begin{bmatrix} -2(3 - 2r)(1 - \xi) & 4(1 - \xi) & -2(1 + 2r)(1 - \xi) \\ 3 - 2r & -2 & 1 + 2r \\ -2(3 - 2r)\xi & 4\xi & -2(1 + 2r)\xi \end{bmatrix} + \pi \left(\frac{b\Delta y}{S}\right) \begin{bmatrix} 3 - 2r & -4 & 2r + 1 \\ 2 & 0 & -2 \\ 2r - 3 & 4 & -2r - 1 \end{bmatrix} \quad (36)$$

The inertial AIC's are

$$[C_{hD^2h}] = \left(\frac{\pi}{12}\right)\left(\frac{b^2\Delta y}{b_r S}\right) \begin{bmatrix} -11 & 4 & -5 \\ 4 & -8 & 4 \\ -5 & 4 & -11 \end{bmatrix} \quad (37)$$

The aerodynamic lag AIC's appear as a column matrix for the strip

$$[C_{hB}] = 2c_{l\alpha}\left(\frac{b\Delta y}{S}\right) \begin{Bmatrix} 2\xi - 2 \\ 1 \\ -2\xi \end{Bmatrix} \quad (38)$$

Since the first two terms in Eq. (26) represent the circulation that determines the aerodynamic lag functions, the interpolation and differentiation matrices are found from these terms and Eq. (34). The interpolation matrix is

$$[H] = \lfloor \tfrac{3}{2} - r \qquad -1 \qquad \tfrac{1}{2} + r \rfloor \quad (39)$$

and the differentiation matrix is

$$[W] = (V/b)\lfloor 1 \qquad -4 \qquad 3 \rfloor \quad (40)$$

In the case of a rigid chord having control points at the one- and three-quarter chord points, the foregoing matrices reduce to the following form

$$[C_{hs}] = 2c_{l\alpha}\left(\frac{b_r\Delta y}{S}\right) \begin{bmatrix} \frac{3}{2} - 2\xi & 2\xi - \frac{3}{2} \\ 2\xi - \frac{1}{2} & \frac{1}{2} - 2\xi \end{bmatrix} \quad (41)$$

$$[C_{hDh}] = c_{l\alpha}\left(\frac{b\Delta y}{S}\right)\begin{bmatrix}(1-r)(4\xi-3) & r(4\xi-3)\\(1-r)(1-4\xi) & r(1-4\xi)\end{bmatrix} + 2\pi\left(\frac{b\Delta y}{S}\right)\begin{bmatrix}1-r & r-1\\ r & -r\end{bmatrix} \quad (42)$$

$$[C_{hD^2h}] = -\left(\frac{\pi}{4}\right)\left(b^2\frac{\Delta y}{b_r S}\right)\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix} \quad (43)$$

$$[C_{hB}] = 2c_{l\alpha}\left(\frac{b\Delta y}{S}\right)\begin{Bmatrix}2\xi-\frac{3}{2}\\ \frac{1}{2}-2\xi\end{Bmatrix} \quad (44)$$

$$[H] = \lfloor 1-r \quad r\rfloor \quad (45)$$

$$[W] = (V/b)\lfloor -1 \quad 1\rfloor \quad (46)$$

Having the various transient AIC's, we are now in a position to formulate the flutter problem as a transient stability problem. We consider the formulation for a system free in space next.

Collocation Formulation of the Flutter Stability Problem

The equation of motion for a vehicle free in space is

$$[M]\{\ddot{h}\} + [C]\{\dot{h}\} + [K]\{h\} = \{F\} \quad (47)$$

where $[M]$ is the mass matrix of the complete system, $[C]$ is the equivalent viscous damping coefficient matrix, $[K]$ is the stiffness matrix of the unrestrained structure, and $\{F\}$ is the applied force matrix. If the only external force is the aerodynamic force due to transient motion, $\{F\}$ is given by Eq. (15). Although a response analysis requires an additional exciting force, the force due to motion is sufficient to formulate the flutter stability problem. We note that both the damping and stiffness matrices are singular for the unrestrained vehicle. Combining Eqs. (15) and (47) leads to

$$[M']\{\ddot{h}\} + [C']\{\dot{h}\} + [K']\{h\} + \tfrac{1}{2}\rho VS\,\Sigma\alpha_n[C_{hB}]\{B_n\} = 0 \quad (48)$$

where

$$[M'] = [M] - \tfrac{1}{2}\rho S\bar{c}[C_{hD^2h}] \quad (49)$$

$$[C'] = [C] - \tfrac{1}{2}\rho VS[C_{hDh}] \quad (50)$$

$$[K'] = [K] - (qS/\bar{c})[C_{hs}] \quad (51)$$

and $\{B_n\}$ is a solution to Eq. (16) for a given value of n. The matrix eigenvalue formulation of the flutter problem is found by combining Eqs. (16) and (48) and the additional transformation equation

$$\{\dot{h}\} = \{w\} \quad (52)$$

into a single first-order matrix differential equation. Choosing two terms in the approximation to the Wagner function leads to two sets of $\{B_n\}$: $\{B_1\}$ and $\{B_2\}$. The first-order matrix differential equation then appears in the partitioned form

$$\begin{bmatrix}M' & C' & 0 & 0\\ 0 & -I & 0 & 0\\ -H & -W & I & 0\\ -H & -W & 0 & I\end{bmatrix}\begin{Bmatrix}\dot{w}\\ \dot{h}\\ \dot{B}_1\\ \dot{B}_2\end{Bmatrix} + \begin{bmatrix}0 & K' & \frac{1}{2}\rho VS\alpha_1 C_{hB} & \frac{1}{2}\rho VS\alpha_2 C_{hB}\\ I & 0 & 0 & 0\\ 0 & 0 & V\beta_1\lceil 1/b\rfloor & 0\\ 0 & 0 & 0 & V\beta_2\lceil 1/b\rfloor\end{bmatrix}\begin{Bmatrix}w\\ h\\ B_1\\ B_2\end{Bmatrix} = 0 \quad (53)$$

or, abbreviating,

$$[A]\{\dot{X}\} + [B]\{X\} = 0 \quad (54)$$

The solution of Eq. (54) has the form

$$\{X\} = \{\bar{X}\}\exp(\gamma t) \quad (55)$$

where $\{\bar{X}\}$ is the eigenvector and γ is the eigenvalue of Eq. (54). Substituting Eq. (55) into Eq. (54) yields

$$(\gamma[A] + [B])\{\bar{X}\} = 0 \quad (56)$$

Before we consider the solution of Eq. (56) for its eigenvalues and eigenvectors, we consider an alternate formulation in terms of structural influence coefficients (SIC's). The foregoing derivation presupposed the availability of the stiffness matrix, whereas frequently the deflection characteristics are given in the form of the flexibility matrix, i.e., the SIC's $[a]$ of the structure restrained in a statically determinate manner. The stiffness matrix can be obtained from theoretical SIC's, e.g., as discussed by Gallagher.[15] However, if the SIC's are derived from ground vibration tests,[16] the stiffness matrix is not defined** because of the singularity of the SIC matrix. We assume that when the SIC's are given for the restrained structure, the matrix of equivalent viscous damping coefficients $[C]$ is still available for the unrestrained condition. In terms of the SIC's, the equation of motion for the unrestrained system is

$$\{h - h_0\} = [a](\{F\} - [M]\{\ddot{h}\} - [C]\{\dot{h}\}) \quad (57)$$

In Eq. (57), $\{h_0\}$ is the displacement of the control points caused by rigid body motion and is given by

$$\{h_0\} = [h_R]\{a_R\} \quad (58)$$

where $[h_R]$ is a matrix of rigid body modes and $\{a_R\}$ is the set of rigid body displacements. The boundary condition on Eq. (57) may be written in terms of the rigid body modal matrix;

$$[h_R]^T(\{F\} - [M]\{\ddot{h}\}) = 0 \quad (59)$$

where, in contrast to Ref. 17, no mass or aerodynamic forces are associated with the support points of the SIC's,†† but all masses and aerodynamic forces are included in $[M]$ and $\{F\}$, respectively. The formulation of the eigenvalue problem in terms of Eqs. (57) and (59) leads to a singular result because $\{a_R\}$ only appears in Eq. (57). In order to avoid a singular formulation, we introduce the deformation mode $\{h_f\}$ which is related to the deflections by

$$\{h_f\} = \{h\} - [h_R]\{a_R\} \quad (60)$$

The eigenvalue problem for this alternate formulation follows by substituting Eqs. (15) and (60) into Eqs. (57) and (59), Eq. (60) into Eq. (16), and by introducing the transformations

$$\{\dot{h}_f\} = \{w_f\} \quad (61)$$

and

$$\{\dot{a}_R\} = \{w_R\} \quad (62)$$

The eigenvalue problem again appears in the form of Eq. (54), but now with

$$\{X\} = \begin{Bmatrix}w_f\\ w_R\\ h_f\\ a_R\\ B_1\\ B_2\end{Bmatrix} \quad (63)$$

** It is discussed later but only in the sense of a generalized inverse.

†† The SIC's must include all loaded points, so that, if there are masses or forces associated with the support points, the SIC's must include the corresponding null elements.

$$[A] = \begin{bmatrix} aM' & aM'h_R & aC' & -\frac{1}{2}\rho VSaC_{hDh}h_R & 0 & 0 \\ h_R{}^TM' & h_R{}^TM'h_R & -\frac{1}{2}\rho VSh_R{}^TC_{hDh} & -\frac{1}{2}\rho VSh_R{}^TC_{hDh}h_R & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ -H & -Hh_R & -W & -Wh_R & I & 0 \\ -H & -Hh_R & -W & -Wh_R & 0 & I \end{bmatrix} \quad (64)$$

$$[B] = \begin{bmatrix} 0 & 0 & I-(qS/\bar{c})aC_{hs} & -(qS/\bar{c})aC_{hs}h_R & \frac{1}{2}\rho VS\alpha_1aC_{hB} & \frac{1}{2}\rho VS\alpha_2aC_{hB} \\ 0 & 0 & -(qS/\bar{c})h_R{}^TC_{hs} & -(qS/\bar{c})h_R{}^TC_{hs}h_R & \frac{1}{2}\rho VS\alpha_1h_R{}^TC_{hB} & \frac{1}{2}\rho VS\alpha_2h_R{}^TC_{hB} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V\beta_1[1/b] & 0 \\ 0 & 0 & 0 & 0 & 0 & V\beta_2[1/b] \end{bmatrix} \quad (65)$$

We consider the solution of the eigenvalue problem in the next section. We note that our transient formulation of the flutter stability problem will also include the rigid body stability and control characteristics, e.g., the short period longitudinal mode; the phugoid mode, of course, cannot be found since a constant forward velocity has been assumed throughout this development.

The Eigenvalue Problem

The choice of a particular technique for solving the eigenvalue problem depends on the distribution of the eigenvalues in the complex plane. The eigenvalues are either real or complex conjugate pairs. The negative real roots correspond to the aerodynamic lags and are approximately given by

$$\gamma \approx -V\beta_1/b \text{ or } -V\beta_2/b \quad (66)$$

A positive real root corresponds to torsional divergence. The complex conjugate roots correspond to the damped vibrations of the system and may be written

$$\gamma \approx \omega(\gamma_s \pm i) \quad (67)$$

where ω is the frequency and γ_s is the decay coefficient of the aeroelastic system. Typical loci of the roots are shown in Fig. 2 as the velocity is increased from zero to the flutter speed. Flutter occurs when the decay coefficient of some mode vanishes.

The power method (the so-called matrix iteration method) is appropriate for finding the eigenvalues of the large unsymmetrical matrix that results from the collocation formulation of the problem. However, difficulties can be anticipated because of the number of close roots that correspond to the aerodynamic lag functions. The eigenvalue subprogram ASC MTRS of Ref. 18 has the capability of finding the complex conjugate roots and of finding close roots occurring in pairs. Since there are as many aerodynamic lag roots as there are strips, the number of close roots will exceed the capability of ASC MTRS. These difficulties may be circumvented by modifying the formulation to permit use of the shifting technique of Wilkinson.[19] If we let

$$\lambda = 1/(\gamma_0 - \gamma) \quad (68)$$

Eq. (56) may be rewritten as

$$\lambda\{\bar{X}\} = ([B] + \gamma_0[A])^{-1}[A]\{\bar{X}\} \quad (69)$$

With the shift γ_0, the power method converges to the eigenvalue λ that corresponds to the value of γ closest to γ_0. From Fig. 2 we see that a generally suitable choice of γ_0 is the imaginary number

$$\gamma_0 = +i\tfrac{1}{2}(\omega_1 + \omega_m) \quad (70a)$$

$$= i\omega_0 \quad (70b)$$

where ω_m is the maximum frequency of interest in the flutter solution. In this way the power method will converge to the vibrational roots and avoid any difficulties with complex conjugate roots or with the close aerodynamic lag roots. However, for generality we consider γ_0 to be complex and given by

$$\gamma_0 = \gamma_1 + i\omega_0 \quad (71)$$

where the shift value of γ_1 will be taken as zero until a specific application suggests a better choice, e.g., when convergence cannot be obtained with a pure imaginary value of γ_0. From the complex eigenvalue λ

$$\lambda = \lambda_R + i\lambda_I \quad (72a)$$

$$= 1/[\gamma_1 + i\omega_0 - \omega(\gamma_s + i)] \quad (72b)$$

we find the frequency

$$\omega = \omega_0 + \lambda_I/(\lambda_R{}^2 + \lambda_I{}^2) \quad (73)$$

and the decay coefficient

$$\gamma_s = (1/\omega)[\gamma_1 - \lambda_R/(\lambda_R{}^2 + \lambda_I{}^2)] \quad (74)$$

If the aerodynamic lag roots are of interest, a real shift is appropriate and ω_0 is chosen as zero. Then from the real eigenvalue λ_R we find

$$\gamma = \gamma_1 - 1/\lambda_R \quad (75)$$

An example of the collocation solution for a restrained system is given in the next section.

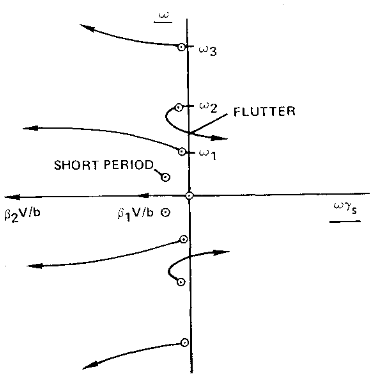

Fig. 2 Root loci of system stability.

Example Flutter Solution for a Restrained Wing

An excellent illustration of the collocation solution of the flutter stability was provided by the jet transport wing analyzed by Bisplinghoff, Ashley, and Halfman[20] throughout their book, and also analyzed by Rodden[21] using the colloca-

Table 1 Strip geometry

Strip No.	b, ft	Δy, ft
1 (inboard)	8.4375	7.7500
2	7.4375	7.4167
3	6.5833	7.5833
4	5.5417	7.9167
5 (tip)	4.6042	7.2500

tion method and the conventional flutter stability analysis. The geometry of the wing is shown in Fig. 3. The flexibility and mass matrices are given in Ref. 21. All structural damping was neglected, i.e., $[C] = 0$, in order to make a direct comparison with the conventional flutter analysis of Ref. 21. The geometry required by the present method consists of the semichord and width of each strip; these are shown in Table 1. The reference semichord $b_r = \bar{c} = 5.468$ ft and the reference wing area $S = 564.236$ ft^2. The analysis was carried out at sea level as before[20,21] where $\rho = 0.002378$ slugs/ft^3. The constants in the exponential approximation to the Wagner function were chosen as those recommended by W. P. Jones[13]: $\alpha_1 = 0.165$, $\beta_1 = 0.041$, $\alpha_2 = 0.335$, and $\beta_2 = 0.320$. The AIC's were calculated for incompressible flow with no aspect ratio correction, i.e., $c_{l_\alpha} = 2\pi$ and $\xi = 0.25$, and by assuming a rigid chord. Although the quarter-chord line has a slight sweep, cosΛ was taken as unity as before. Based on the above assumptions, the partitions of the AIC's, the interpolation matrix and the differentiation matrix, were derived from Eqs. (41–46).

For the calculation of the eigenvalues we regarded four modes of interest. Since the previous vibration analysis had shown $\omega_1 = 12.798$, $\omega_2 = 22.322$, $\omega_3 = 45.744$, and $\omega_4 = 73.504$ rad/sec, we chose the shift frequency $\omega_0 \approx (\omega_1 + \omega_4)/2 \approx 43$ rad/sec to begin the calculations. The results for the frequencies are shown in Fig. 4 and for the damping of the unstable modes in Fig. 5; in Fig. 5, $2\gamma_s$ is compared with the artificial required structural damping g of the conventional oscillatory flutter solution.

For velocities less than 1100 fps the initial choice of $\omega_0 = 43.0$ yielded the first 4 modes without difficulty, and at velocities greater than 1400 fps, it yielded the 2nd through the 5th modes without difficulty. The closeness of the 4th and 5th mode frequencies near $V = 1200$ fps prevented convergence, but the shift frequency $\omega_0 = 70.0$ led to the solution. At this point it became apparent that the earlier solution of Ref. 21 was incorrect to the extent that it described one of the flutter modes as the 4th mode, whereas it is actually the 5th mode that flutters while the 4th mode always remains stable. A shift to $\omega_0 = 85.0$ at velocities up to 1000 fps showed that no higher mode was of any practical interest.

We next sought to follow the first bending mode behavior at velocities above 1000 fps. It was anticipated that the frequency of the first mode would go to zero and the first mode would become the torsional divergence mode. The divergence speed was determined by Ref. 22 to be 1419.9 fps. However, it was discovered in developing the first-mode frequency curve (using a number of complex shifts, $\gamma_0 = \gamma_1 + i\omega_0$) that the frequency decreased rapidly at speeds slightly higher than the divergence speed. The frequency went to zero in a small range of velocity, $1431.0 < V < 1432.5$ fps, and then increased rapidly as shown in Fig. 4. No convergence difficulties were encountered as the first-mode frequency curve crossed that of the second mode because of the large difference in the damping of each mode.

Since the divergence speed was not found by following the mechanical roots of the system, the aerodynamic lag roots were next investigated. The roots corresponding to $\beta_1 =$

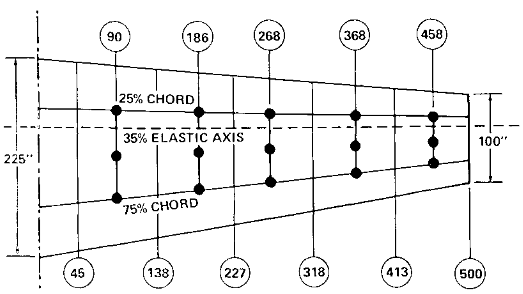

Fig. 3 Jet transport wing geometry.

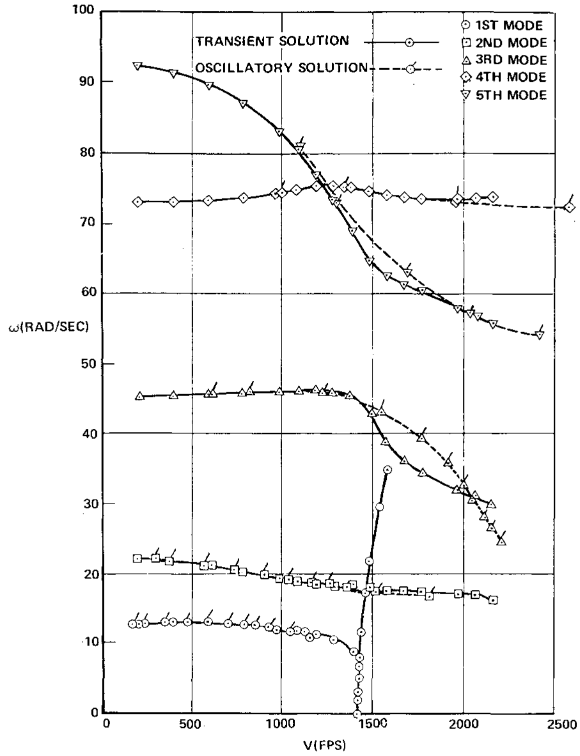

Fig. 4 Comparison of frequencies.

0.041 are shown in Fig. 6; a number of real shifts ($\gamma_1 = -1.0$, -4.5, and -6.0) led to the solution. The approximation of Eq. (66) for the lag roots was found to be reasonably accurate except for the divergence root as the divergence speed is approached. The divergence speed was found by interpolation in Fig. 6 to be in accord with the classical eigenvalue divergence solution.

Although this example is highly academic, since incompressible flow can hardly be assumed for the range of speeds considered and flutter at speeds above divergence cannot be a practical concern, it does serve to illustrate the manner of solution of both the static and dynamic aeroelastic stability problems by the transient method. The flutter speeds

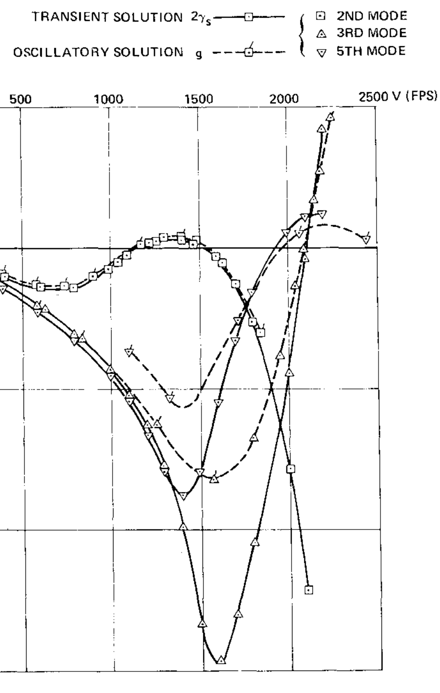

Fig. 5 Comparison of damping for unstable modes.

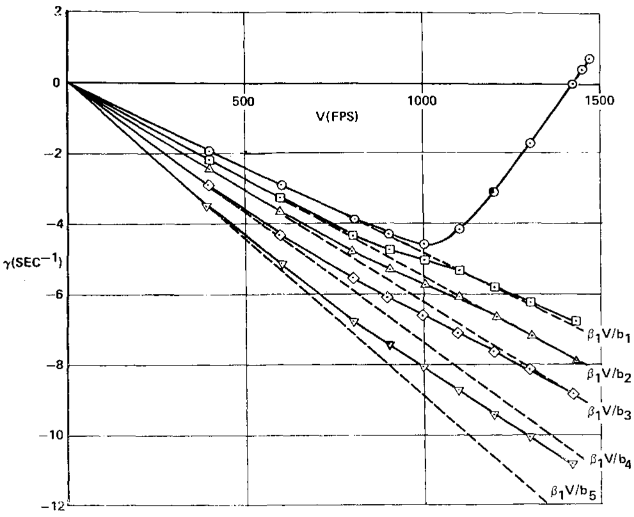

Fig. 6 Aerodynamic lag roots for $\beta_1 = 0.041$.

are found to agree with those found by the conventional stability analysis to the accuracy expected from the relationship between the approximation to the Wagner function and the Theodorsen function. The comparison of damping curves shows up some rather interesting agreements as well as disagreements: in the first flutter mode (the 2nd vibration mode) the true and artificial damping agree closely for the velocities of interest, whereas in the second and third flutter modes (the 3rd and 5th vibration modes) the true dampings differ substantially from the artificial dampings except at low velocities. The large difference in damping between the two solutions for the 3rd mode and also for the 5th mode was anticipated and this example was expected to illustrate this feature as a difference between the conventional and the transient solutions of the flutter problem. It should be remarked that the methods of Refs. 1–7 are applicable to this example because the frequency-velocity and damping-velocity curves are well behaved. Frueh[23] has applied the methods of Refs. 1–7 to this example and has shown good agreement with the damping predictions of the present method.

Modal Solution of the Flutter Stability Problem

The use of SIC's and AIC's in the collocation flutter analysis provides a number of advantages in terms of accuracy and convenience. However, computing time or computer capacity may limit the analysis of large systems and a modal solution must be used. In order to formulate the modal solution of Eq. (48), it is necessary to eliminate the aerodynamic lag functions, inasmuch as there is no basis for choosing aerodynamic lag "modes." The elimination of the lag functions is conveniently accomplished by Laplace transform methods. We rewrite Eq. (48) by introducing the solution for the lag functions from Eq. (22) in matrix form. Equation (48) becomes

$$[M']\{\ddot{h}\} + [C']\{\dot{h}\} + [K']\{h\} + \tfrac{1}{2}\rho VS\Sigma\alpha_n[C_{hB}] \times \left\lceil \exp\left(-\beta_n \frac{Vt}{b}\right)\right\rfloor \int_0^t \left\lceil \exp\left(\beta_n \frac{V\tau}{b}\right)\right\rfloor \{Q'\}d\tau = 0 \quad (76)$$

Taking a two term exponential approximation for the Wagner function, we find the Laplace transform of Eq. (76) to be

$$(s^2[M'] + s[C'] + [K'])\{\mathcal{L}(h)\} + \tfrac{1}{2}\rho VS[C_{hB}](\alpha_1\lfloor 1/(s + \beta_1 V/b)\rfloor + \alpha_2\lfloor 1/(s + \beta_2 V/b)\rfloor)\{s\mathcal{L}(Q)\} = 0 \quad (77)$$

assuming the system to be initially at rest. The format of $[C_{hB}]$ permits certain matrix manipulations. These are justified because of the strip theory assumption but are not valid in general. We define a premultiplying matrix $\lfloor 1/(s + \beta V/b)\rfloor^E$, such that

$$\lfloor 1/(s + \beta V/b)\rfloor^E[C_{hB}] = [C_{hB}]\lfloor 1/(s + \beta V/b)\rfloor \quad (78)$$

in which the premultiplier of the left-hand side of Eq. (78) is an appropriately expanded form of the postmultiplier of the right-hand side; the superscript E denotes the expansion operation. If we perform the expansion and interchange operations on Eq. (77) and premultiply the resulting equation by $\lfloor s + \beta_1 V/b\rfloor^E\lfloor s + \beta_2 V/b\rfloor^E$, we obtain the Laplace transform of the equations of motion with the aerodynamic lag functions eliminated.

$$\lfloor s + \beta_1 V/b\rfloor^E\lfloor s + \beta_2 V/b\rfloor^E(s^2[M'] + s[C'] + [K'])\{\mathcal{L}(h)\} + \tfrac{1}{2}\rho VS(\alpha_1\lfloor s + \beta_2 V/b\rfloor^E + \alpha_2\lfloor s + \beta_1 V/b\rfloor^E)[C_{hB}]\{s\mathcal{L}(Q)\} = 0 \quad (79)$$

Inverting the transform leads to the desired equation of motion.

$$[M']\{h^{IV}\} + ([C'] + (\beta_1 + \beta_2)V\lfloor 1/b\rfloor^E[M'])\{\dddot{h}\} + ([K'] + (\beta_1 + \beta_2)V\lfloor 1/b\rfloor^E[C'] + \beta_1\beta_2V^2\lfloor 1/b^2\rfloor^E \times [M'])\{\ddot{h}\} + ((\beta_1 + \beta_2)V\lfloor 1/b\rfloor^E[K'] + \beta_1\beta_2V^2\lfloor 1/b^2\rfloor^E \times [C'])\{\dot{h}\} + \beta_1\beta_2V^2\lfloor 1/b^2\rfloor^E[K']\{h\} + \tfrac{1}{2}\rho VS(\alpha_1 + \alpha_2) \times [C_{hB}]\{\ddot{Q}\} + \tfrac{1}{2}\rho VS(\alpha_1\beta_2 + \alpha_2\beta_1)V\lfloor 1/b\rfloor^E[C_{hB}]\{\dot{Q}\} = 0 \quad (80)$$

We note that the circulation matrix $\{Q\}$ is related to the deflections through Eq. (16). The order of the differential equation of motion is $(2 + n)$ where n is the number of terms in the approximation to the Wagner function.

The modal solution to Eq. (80) follows by expressing the deflections as a series of modes

$$\{h\} = [h_n]\{a_n\} \quad (81)$$

where the $\{a_n\}$ are the generalized coordinates and $[h_n]$ includes both rigid body and vibration modes, and by applying the Galerkin method through premultiplication by $[h_n]^T$. The resulting equation need not be shown. The eigenvalue problem follows by transforming the modal equation of motion into an equivalent system of first-order equations. The eigenvalue problem is of order $(2 + n)N$, where N is the number of modes in $[h_n]$.

It should be noted that this modal method requires the stiffness matrix for its formulation; an analogous modal method using SIC's has not been found. When a purely theoretical analysis is made, this requirement poses no difficulties. However, when ground vibration test modes are used, the SIC's found from the method of Ref. 16 are singular, and the stiffness matrix can only be constructed by means of the methods of generalized inverses as suggested by Stahl.[24] Although the stiffness matrix derived by this technique is not the actual stiffness matrix of the structure, since convergence to the actual stiffness requires all of the modes of the system, it does contain all of the experimental data and is therefore consistent with the usual limitations of modal formulations.

Concluding Remarks

A strip method has been developed for the prediction of subcritical frequency and damping characteristics for guidance of subsonic wind tunnel and flight flutter tests. A series of transient AIC's have been derived for a cambering airfoil that includes a newly defined matrix of aerodynamic lag AIC's. The AIC's, although based on incompressible two-dimensional flow, have been modified for the effects of sweep, aspect ratio, and Mach number by utilizing aerodynamic parameters as determined from any suitable steady flow theory or from static wind tunnel tests. Such modifications of the aerodynamic parameters on each strip allow application of the method at all subsonic speeds; the method is not applicable at supersonic speeds because the transient flow mechanism is different, i.e., the exponential approximation to the Wagner function is not valid.

Collocation formulations of the subcritical flutter problem have been given using both stiffness and flexibility matrices as the basic structural data. The eigenvalue problem has been discussed and a special technique for its solution has been reviewed. The eigenvalue solution has been illustrated for an example of a restrained (cantilevered) wing with 5 strips and 10 elastic degrees of freedom. A specialized modal formulation has also been given.

Although no control surface has been included in the present analysis, the extension to its inclusion is straightforward. The oscillatory aerodynamic coefficients for control surface degrees of freedom are well known, but coefficients that include coupling with camber have only been obtained[25] for control surface rotation about its leading edge. The general case of the aerodynamically balanced control surface requires the additional coefficients for the coupling between the camber and control surface plunging. These may readily be derived by the method of Küssner[26] as employed by Spielberg[10] and Tyler.[25]

It should be recognized that the computations involved in this method are considerably more extensive than those required by the conventional technique for flutter analysis. It is not expected, and it is not desirable, that the present procedure would replace the conventional one. Its general use should be limited to prediction of the behavior of wind tunnel and flight flutter test configurations before the tests are conducted and to monitor the measurements while the tests are in progress. The normal design and contractually required flutter analyses should be carried out by the conventional method, although the present method may occasionally be useful in explaining peculiarities in the shape of required damping-velocity stability curves.

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